



Fracture Mechanics

Soheil Mohammadi
School of Civil Engineering
University of Tehran

Tel: +98-21-6111 2258, Fax: +98-21-6640 3808, Cell: 0912-5030 392

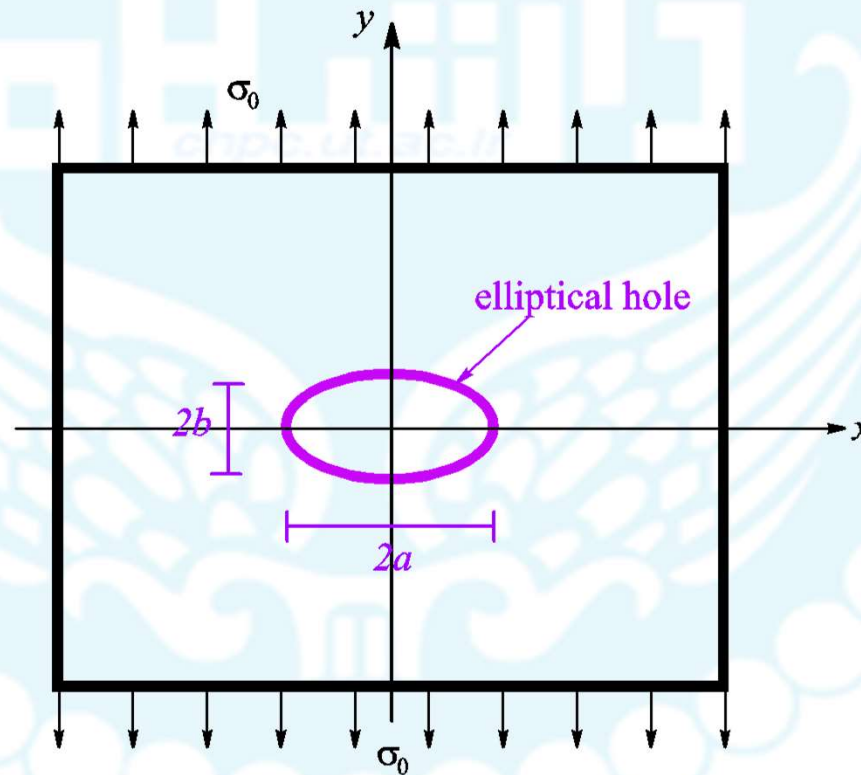
Email: smoham@ut.ac.ir, Homepage: <http://chpc.ut.ac.ir/smohammadi>
Facebook: [prof.soheil.mohammadi](https://www.facebook.com/prof.soheil.mohammadi)

The background features a large, faint, light blue watermark of the University of Mohammedi Seal. The seal is circular with a scalloped outer edge. Inside the circle, there is a central emblem with Arabic calligraphy. The words "UNIVERSITY OF MOHAMMEDI" are written in a circular path around the central emblem, with each letter in its own small circle.

Infinite Tensile Plate with an Ellipsoidal Hole

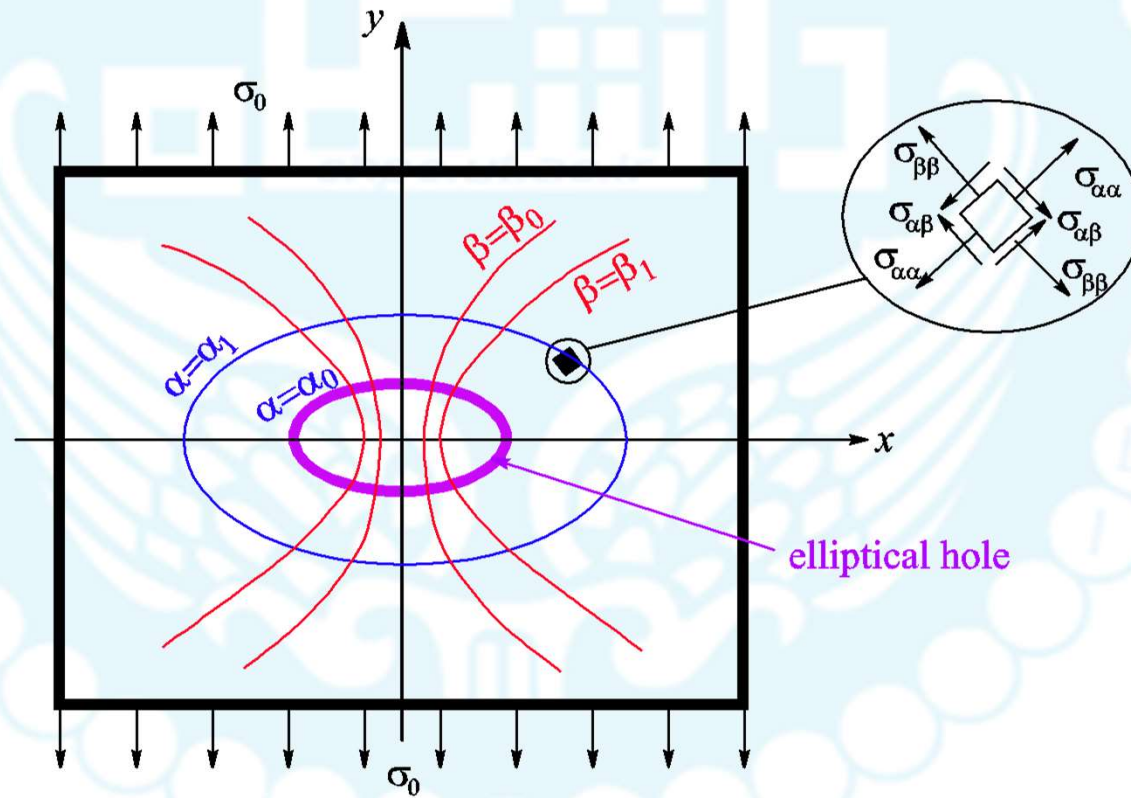
Infinite tensile plate with an ellipsoidal hole

Inglis (1913), a professor of naval architecture, independent of an earlier work by Kolosov, solved the problem of stress concentration around an elliptical hole in an infinite plate subjected to uniform stress loading.



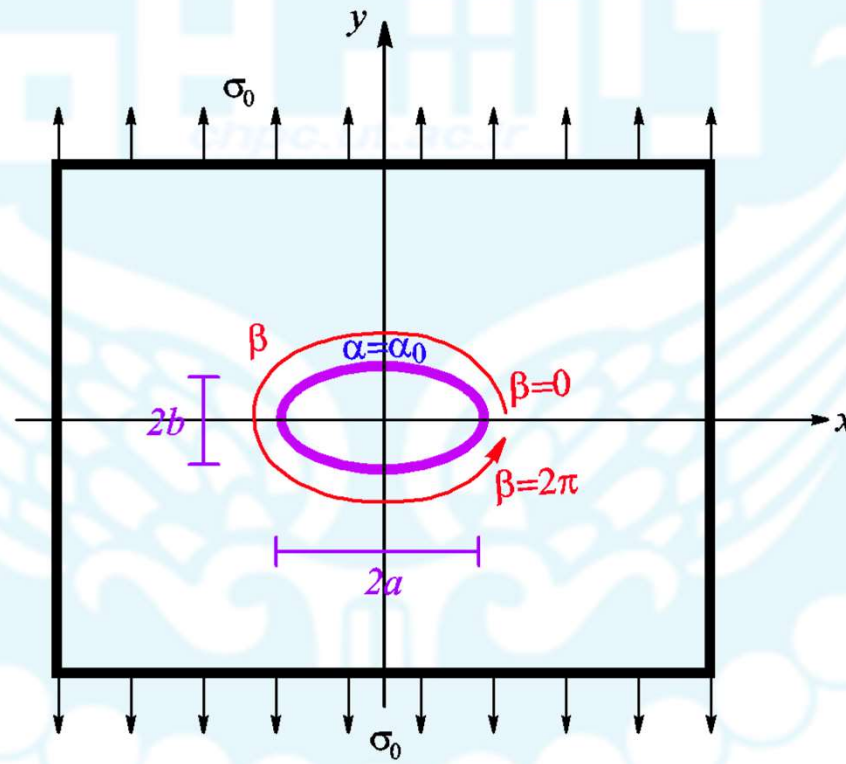
Infinite tensile plate with an ellipsoidal hole

The problem is difficult to analyse in a Cartesian coordinate system. The boundary conditions around the ellipsoidal hole can be better expressed in a curvilinear coordinate systems.



Infinite tensile plate with an ellipsoidal hole

Adopting the curvilinear coordinate system, we define a and b as the major and minor semi-axes respectively. The elliptical hole is itself defined along $\alpha = \alpha_0$, and as we go around the ellipse β varies from 0 to 2π .



Infinite tensile plate with an ellipsoidal hole

Then:

$$x = c \cosh \alpha \cos \beta$$

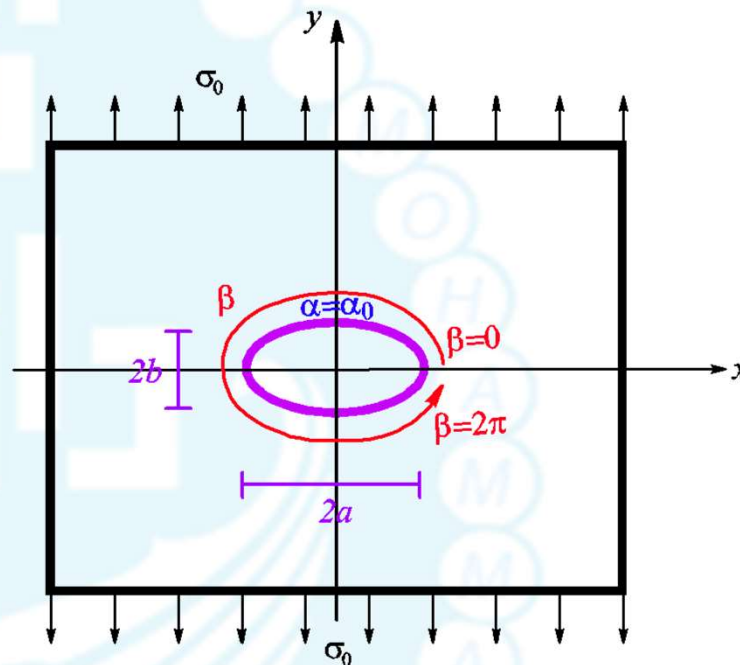
$$x \Big|_{\substack{\alpha=\alpha_0 \\ \beta=0}} = a \rightarrow a = c \cosh \alpha_0 \cos 0$$

$$\rightarrow a = c \cosh \alpha_0$$

$$y = c \sinh \alpha \sin \beta$$

$$y \Big|_{\substack{\alpha=\alpha_0 \\ \beta=\pi/2}} = b \rightarrow b = c \sinh \alpha_0 \cos(\pi / 2)$$

$$\rightarrow b = c \sinh \alpha_0$$



Two sets of boundary conditions should be satisfied:

At infinity;

$$\sigma_{yy} = \sigma_0$$

$$\sigma_{xx} = \sigma_{xy} = 0$$

Recalling:

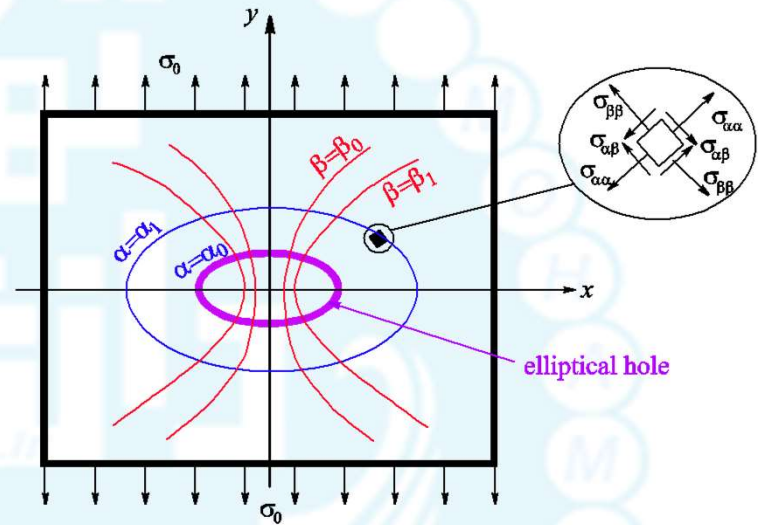
$$\sigma_{xx} + \sigma_{yy} = 4 \operatorname{Re} \psi'(z)$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2[\bar{z}\psi''(z) + \chi''(z)]$$

Then, the complex functions ψ and χ must satisfy:

$$4 \operatorname{Re} \psi'(z) = \sigma_0$$

$$2[\bar{z}\psi''(z) + \chi''(z)] = \sigma_0$$

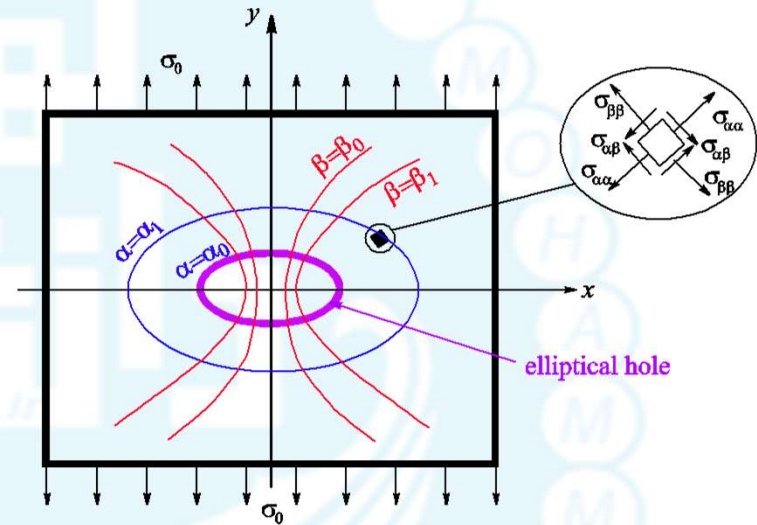


Two sets of boundary conditions should be satisfied:

Around the elliptical hole ($\alpha=\alpha_0$):

$$\sigma_{\alpha\alpha} = 0$$

$$\sigma_{\alpha\beta} = 0$$



Infinite tensile plate with an ellipsoidal hole

The following complex stress potential functions, proposed by Inglis in the curvilinear coordinate system α and β , satisfy the boundary conditions and are periodic in β :

$$\Phi = \operatorname{Re}[\bar{z}\psi(z) + \chi(z)]$$

$$\psi(z) = \frac{1}{4}\sigma_0 c \left[\left(1 + e^{2\alpha_0}\right) \sinh p - e^{2\alpha_0} \cosh p \right]$$

$$\chi(z) = -\frac{1}{4}\sigma_0 c^2 \left[\left(\cosh 2\alpha_0 - \cosh \pi \right) p + \frac{1}{2} e^{2\alpha_0} \right]$$

$$p = \alpha + i\beta - \cosh 2 \left(p - \alpha_0 - i \frac{\pi}{2} \right)$$

From the relations of stress components in the curvilinear system:

$$\sigma_{\alpha\alpha} + \sigma_{\beta\beta} = 2[\psi'(z) + \bar{\psi}'(\bar{z})] = 4\operatorname{Re}\psi'(z)$$

$$\sigma_{\beta\beta} - \sigma_{\alpha\alpha} + 2i\sigma_{\alpha\beta} = 2e^{2i\theta} [\bar{z}\psi''(z) + \chi''(z)]$$

$\sigma_{\beta\beta}$ on the hole boundary can be computed:

$$\sigma_{\beta\beta}\Big|_{\alpha=\alpha_0} = 2[\psi'(z) + \bar{\psi}'(\bar{z})]$$

$$\sigma_{\beta\beta}\Big|_{\alpha=\alpha_0} = \sigma_0 e^{2\alpha_0} \left[\frac{\sinh 2\alpha_0 (1 + e^{-2\alpha_0})}{\cosh 2\alpha_0 - \cos 2\beta} - 1 \right]$$

The maximum value of $\sigma_{\beta\beta}$ occurs at the end points of ellipse:

$$\sigma_{\beta\beta} \Big|_{\substack{\alpha=\alpha_0 \\ \beta=0,\pi}} = \frac{\sinh 2\alpha_0 - 1 + e^{2\alpha_0}}{\cosh 2\alpha_0 - 1} \sigma_0 \quad [= \sigma_{22}]$$

Note that at these points $\sigma_{\beta\beta} = \sigma_{yy}$.

This is rather difficult to compute because it is in terms of curvilinear coordinates. Combining:

$$\left. \begin{aligned} a &= c \cosh \alpha_0 \\ b &= c \sinh \alpha_0 \\ \cosh^2 \alpha_0 - \sinh^2 \alpha_0 &= 1 \end{aligned} \right\} \rightarrow c^2 = a^2 - b^2$$

and using mathematical formulations:

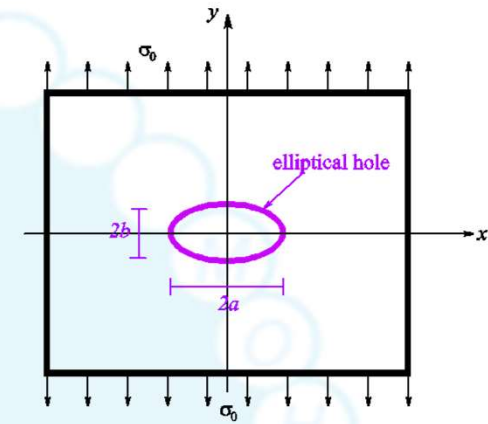
$$\left. \begin{aligned} \cosh 2\alpha_0 &= 2 \cosh^2 \alpha_0 - 1 \\ \sinh 2\alpha_0 &= 2 \sinh \alpha_0 \cosh \alpha_0 \end{aligned} \right\} \rightarrow \begin{cases} \sinh 2\alpha_0 = \frac{2ab}{c^2} \\ \cosh 2\alpha_0 = \frac{a^2 + b^2}{c^2} \end{cases}$$

leads to:

$$\sigma_{yy} \Big|_{\substack{\alpha=\alpha_0 \\ \beta=0,\pi}} = \sigma_{\beta\beta} \Big|_{\substack{\alpha=\alpha_0 \\ \beta=0,\pi}} = \sigma_0 \left(1 + 2 \frac{a}{b} \right)$$

From the final solution:

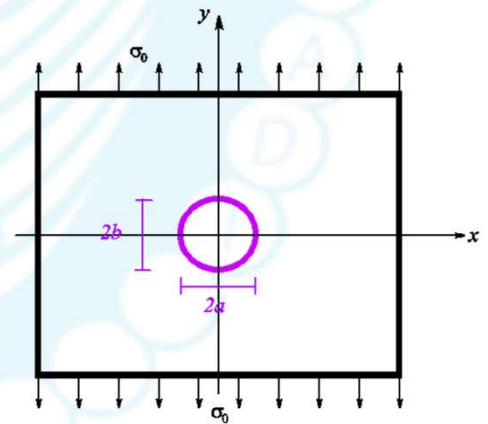
$$\sigma_{yy} \bigg|_{\substack{\alpha=\alpha_0 \\ \beta=0,\pi}} = \sigma_{\beta\beta} \bigg|_{\substack{\alpha=\alpha_0 \\ \beta=0,\pi}} = \sigma_0 \left(1 + 2 \frac{a}{b} \right)$$



It is observed that:

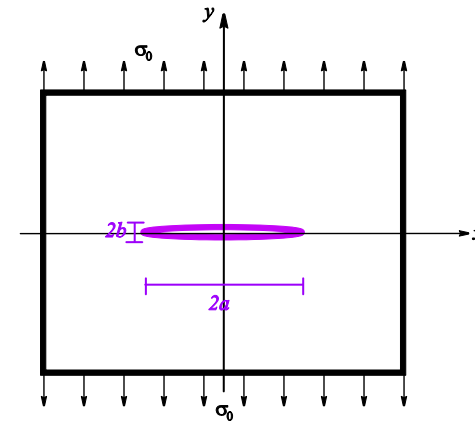
1. For $a=b$, the stress concentration factor of 3 is recovered, which is similar to the solution of a circular hole.

$$a = b \quad : \quad \sigma_{yy} \bigg|_{\substack{\alpha=\alpha_0 \\ \beta=0,\pi}} = 3\sigma_0$$



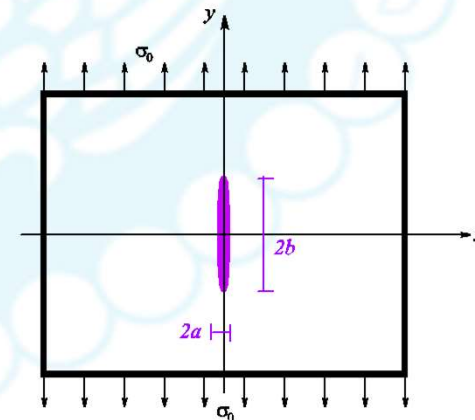
2. For $a/b \rightarrow \infty$, or a degenerated ellipse (similar to a crack shape), infinite stress is obtained:

$$\frac{a}{b} \rightarrow \infty : \sigma_{yy} \Big|_{\substack{\alpha=\alpha_0 \\ \beta=0,\pi}} = \infty$$



3. For a crack directed along the applied tensile stress ($a/b=0$), however, the stress at the crack tip remains at the finite value σ_0 . In fact, the stress state remains similar to an uncracked tensile plate.

$$\frac{a}{b} \rightarrow 0 : \sigma_{yy} \Big|_{\alpha=\alpha_0} = \sigma_0$$



4. The solution can also be written in terms of the radius of curvature ρ ,

$$\sigma_{yy} \Big|_{\substack{\alpha=\alpha_0 \\ \beta=0,\pi}} = \sigma_{\beta\beta} \Big|_{\substack{\alpha=\alpha_0 \\ \beta=0,\pi}} = \sigma_0 \left(1 + 2\sqrt{\frac{a}{\rho}} \right)$$

5. The second term is larger than the first term, showing that the stress concentration factor is inversely proportional to the radius of curvature of an opening/defect. As a result, sharper openings are expected to generate higher SCFs.

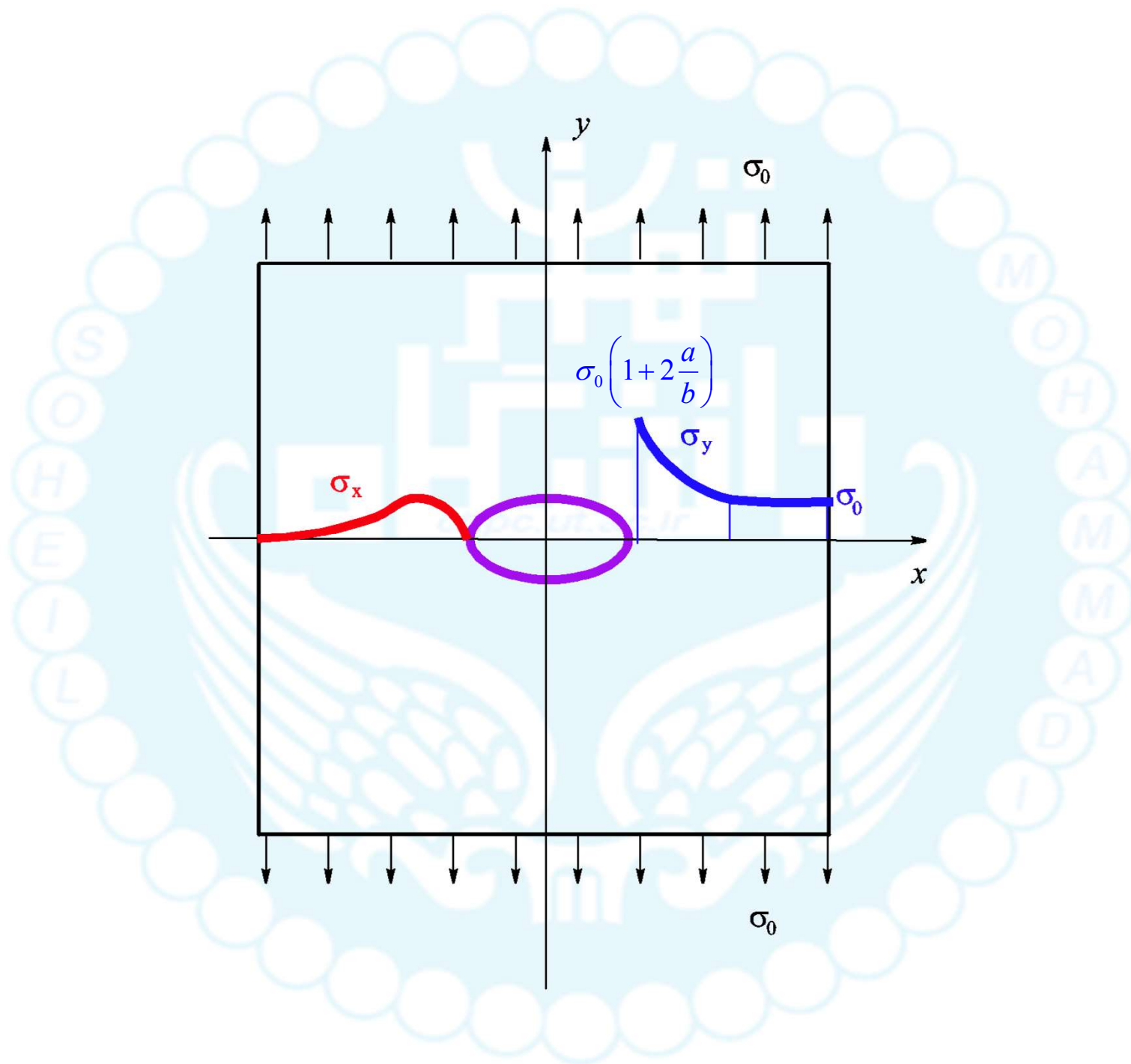
$$\sigma_{yy} \Big|_{\substack{\alpha=\alpha_0 \\ \beta=0,\pi}} \propto \sigma_0 \sqrt{\frac{a}{\rho}}$$

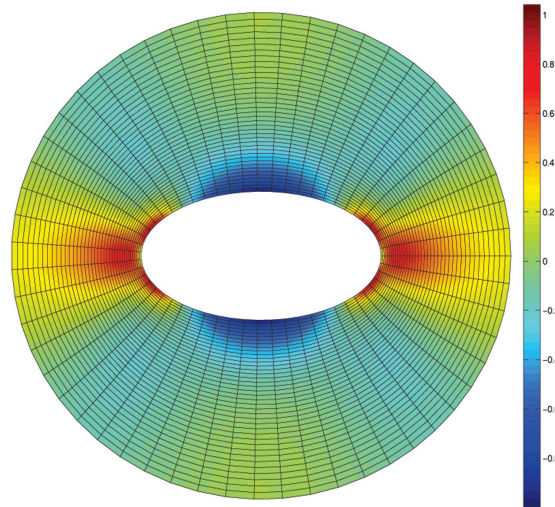
6. In limit, at the tip of a sharp crack (with zero radius of curvature), the SCF becomes infinite.

$$\rho \approx 0 \quad : \quad \sigma_{yy} \Big|_{\substack{\alpha=\alpha_0 \\ \beta=0,\pi}} \rightarrow \infty$$

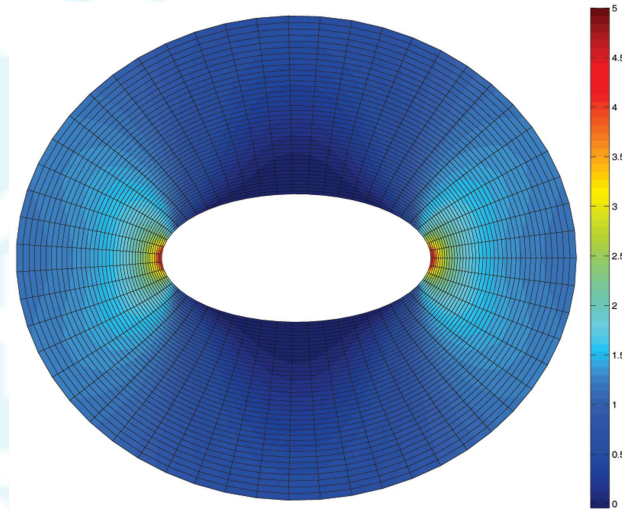
7. While this equation is obtained specifically for a single point on an elliptical hole in an infinite tensile plate, this concept is usually adopted by the fracture mechanics engineers for all fracture problems (even with no general analytical derivation).

$$\sigma_{yy} \Big|_{\substack{\alpha=\alpha_0 \\ \beta=0,\pi}} \propto \sigma_0 \sqrt{\frac{a}{\rho}}$$

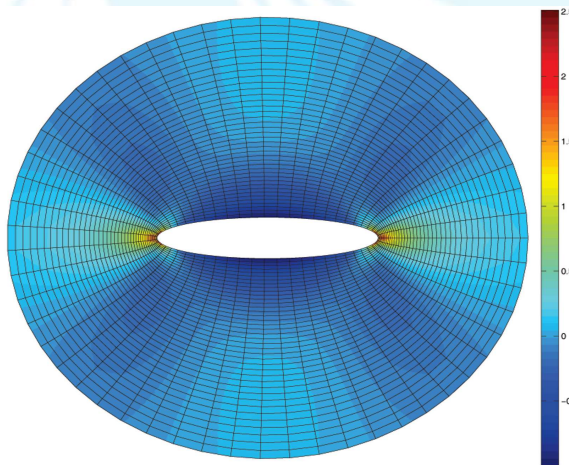




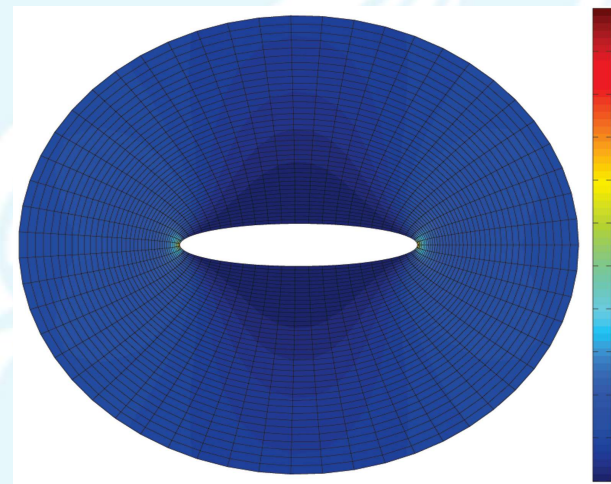
$$\sigma_{xx} (a/b = 2)$$



$$\sigma_{yy} (a/b = 2)$$



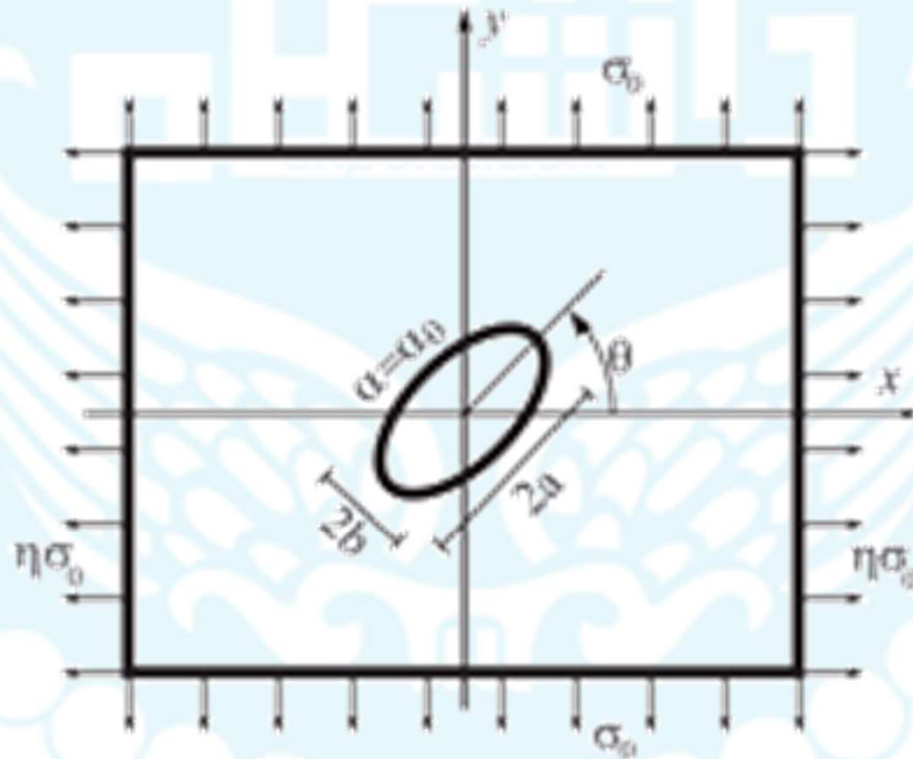
$$\sigma_{xx} (a/b = 5)$$



$$\sigma_{yy} (a/b = 5)$$

For a more general form of the problem, which is an inclined elliptical hole in an infinite biaxial, Murrell 1964 obtained the following solution:

$$\sigma_{\beta\beta}\big|_{\alpha=\alpha_0} = \sigma_0 \left[\frac{(1+\eta) \sinh 2\alpha_0 - (1-\eta) (\cos 2\theta - e^{2\alpha_0} \cos 2(\beta - \theta))}{\cosh 2\alpha_0 - \cos 2\beta} \right]$$



HW P2-C03#1:

Draw the stress contours for an infinite tensile plate with an ellipsoidal hole, with various a/b ratios (0.1,0.5,1,10):

- 1- in the (α, β) curvilinear system
- 2- in the (x, y) Cartesian system
- 3- Compare the results of $a/b=1$ with the circular hole.

